Mathematical Modeling of Nonlinear Dynamic Processes in Information Technologies of Mathematics

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Abstract—**The variety of unsolved mathematical problems is analyzed. It is shown that unsolved problems in the theory of numbers severely limit the further development of mathematical science in both theoretical and applied aspects. The foundations of experimental mathematics are formed as a tool for constructing information technologies in pure and applied number theory. On the basis of Artin's hypothesis, a generalized hypothesis is formed and shown as a means of non-linear dynamical processes to obtain a sufficiently accurate solution of it.**

Index Terms—**Gödel's theory; groups of residues; primitive root; distribution of prime numbers; recursion; Artin's hypothesis; estimation of Artin parameters.**

I. Introduction

In mathematics, until the thirties of the twentieth century, there was a belief that any problem in mathematics could be solved. However, in 1935 Gödel proved that if constructive mathematical theory includes arithmetic, then it always contains a true theorem, which is unprovable by means of the given mathematical theory [1]. Since this moment, active research has begun in the theory of recursive functions and effective computability [2]. In parallel, many unsolved mathematical problems have become the object of detailed research in terms of assessing the complexity of their solution. At the present time, a large number of mathematical problems are known about which there is no information on their solvability. In the field of modern number theory a large list of such problems with detailed analysis is given in monographs [3, 4] and a number of other papers. One of these problems is the Artin hypothesis [5] formed in 1927 and has not been solved so far.

An important problem in number theory is the description of the law of distribution of prime numbers. This problem was solved by Hadamard and Valle-Poussin, independently of each other, in 1896 [6]. They proved that the number of prime numbers $\pi(x)$ less than or equal to x is determined by the expression

$$\pi(x) = \int_2^x \frac{dt}{\ln t} + O\left(x \cdot e^{-\frac{c}{2}\sqrt{\ln x}} \right) \tag{1}$$

where c is an absolute constant. This analytically proved form of representation of the law of distribution of prime numbers has already become universally recognized in the mathematical world. Yet two things should be noted. First, it was obtained on the basis of the analytic zeta-Riemann function, which, until it is proved, adequately describes the distribution of primes in a complex space. According to the Riemann hypothesis, all the zeros of the zeta function are on the line passing through the point equal to $1/2$. This millennium hypothesis has not yet been proved. And this fact is the basis for criticizing all the results obtained on the basis of the zeta-Riemann function.

The second circumstance is that simultaneously with this fact the dynamics of the change of $O\left(x \cdot e^{-\frac{c}{2}\sqrt{\ln x}} \right)$ [7] is investigated. In [8, 9], an estimate of the entropy of this estimate is obtained and it is proved that it is fractal in nature. These facts are the basis for the formation of proposals on the need to study other models for the distribution of prime numbers. Some results of such investigations are given in monographs [10, 11]. Another problem related to the distribution of prime numbers appeared in 1927, when the well-known mathematician Artin formed a hypothesis about the distribution of prime numbers for which the natural number $a > 1$ is given is their primitive root [12, 13].

According to the Artin conjecture [13], the set of such prime numbers has the distribution law $\pi(x, a)$ in the form of the expression

$$\pi(x, a) = c(a) \cdot \pi(x) \tag{2}$$

where $\pi(x)$ is the distribution of primes, and $c(a)$ is a constant depending on a. So far, despite numerous studies, this hypothesis has not been resolved. At the same time, it is not known whether this is true for any values of a. If the hypothesis is correct, the question remains how to evaluate the constant $c(a)$ for each particular a and what properties of the number a affect its value. Answers to these questions are still lacking. In [14, 15] a detailed analysis of all research results in the field of the Artin hypothesis solution is given.

It should be noted that the proof of the Artin conjecture is important both from the theoretical point of view in number theory and from the application point of view, since its positive solution is important in cryptography, coding theory, and the theory of dynamical systems. In [16], a generalized Artin conjecture was formulated for any $a > 1$, i.e. a may not be a primitive root. According to Artin's generalized theory, equality is valid

$$\pi(x,a,i) = c(a,i) \cdot \pi(x) \tag{3}$$

where $a > 1$, i is the index of the class of primes in the classification of prime numbers generated by the numbers a, $c(a,i)$ is a constant. According to the classification constructed in [16]

$$P_{a,i} = \left\{ p \in P \mid \frac{(p-1)}{card_a(p)} = i \right\} \tag{4}$$

where $card_a(p)$ is the length of the recursion $x_{n+1} \equiv ax_n(\mathrm{mod}\, p)$ for $x_0 = 1$.

It is not difficult to show that for any $a > 1$ the equality

$$\sum_{i=1}^{\infty} c(a,i) = 1 \tag{5}$$

This means that prime numbers are uniformly distributed in classes $P_{a.i}$ for any a. By uniformity it is meant that within each class of primes $P_{a.i}$ the logarithmic law of distribution of primes is preserved. The constant $c(a,i)$ defines the measure of the uniform decimation of primes based on the value of a. If $i-1$ then a is the primitive root of all primes $P_{a.1}$.

The determination of $c(a,i)$ for any a, i by analytical methods is unlikely in the short term. However, the formation and development of experimental mathematics [17, 18] opens another way to solve this problem by computer modeling methods for nonlinear dynamic processes of prime numbers formation.

II. SIMULATION OF DYNAMIC PROCESSES OF DISTRIBUTION OF SIMPLE NUMBERS IN THE SYSTEM OF CLASSIFICATION BY THE MODULE OF NATURAL NUMBERS

The process of modeling the dynamics of the formation of prime numbers was built on the following assumptions. Suppose that we are given an ordered set of prime numbers $P = \{p_1, p_2, \ldots, p_k, \ldots\}$ whose elements are ordered in ascending order. All this set was broken into a subset of 500,000 prime numbers. The number 500000 is due to the limitations of MS Excel, as a tool for statistical analysis, a number of characteristics of the process of forming prime numbers. Only one limitation is important. Always choose 500000 consecutive prime numbers from the set P. In the modern version of Excel, this number can be increased to one million. If you use a powerful computer, you can choose any larger number instead of a million.

The realized variant of the study of dynamic processes of the formation of prime numbers includes the following indicators: the number of the prime number p in the ordered set P, the value of the prime number p, the length of the recursion of the numbers $card_a(p)$ for the same value a for all primes P, the value $ind_a(p)$ of the index of the class, t .e. $ind_a(p) = (p-1)/card_a(p)$, the residue values modulo any natural module $n > 1$, for all classes and any other analyzed properties of primes or factors of the decomposition of the number $p-1$ into prime factors. To each simple factor p_i in the decomposition $p-1 = \prod_{i=1}^{n} p_i^{\alpha_i}$, one indicator of the dynamic process of the formation of prime numbers is put in correspondence, individual values that can be analyzed for any other indicators are the values for them of the residue modulo the natural number $n > 1$. The only exception is $ind_a(p)$. The number of controlled indicators analyzed in the Excel environment can be expanded.

The basis for the emergence and development of experimental mathematics is the following iterative scheme for solving mathematical problems:

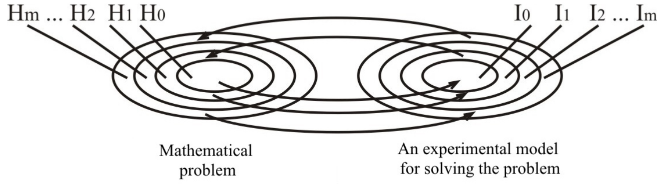

Figure 1. Scheme of the dynamic process of information technology formation to solve the problem

According to the idea of experimental mathematics, at the first iteration we start from hypothetically known data. But it is also the basis for obtaining experimental information on the basis by which analytical methods of number theory give an extended representation of the hypothesis in the form of H_i. It is possible that the hypothesis can be corrected or even rejected as not true. Refined from the point of view of information technology in mathematics, the hypothesis H_i is used to develop from the point of view of deepening experimental mathematics the model of in-depth studies at the I_1 level.

The iteration process is continued until an analytically valid solution of the generated hypothesis is obtained. Since the generalized Artin conjecture is considered in this paper, we present the results of estimating the constant $c(a,i)$ for the case $a = 4$ and $i = 2$. The number $a = 4$ is a perfect square, and therefore it cannot be a primitive root. From the point of view of the generalized Artin conjecture, this is as interesting and important as in the case when $a = 4$ is a primitive root.

Based on the data given in [16], estimates of $c(4,i)$ for $i = 2,4,...,20,..$ were obtained. It is shown that their values are stable for class $P_{4.2}$. class with $ind_4(p) = 2$ to the fourth decimal place. They are presented in the table 1 and table 2.

Similar results were obtained for $a = 2,3,4,5...,16$. The results for the primitive roots 2, 3, 5, 7, 8, 10, 11, 12, 13, 14, 15 are shown in table 3.

TABLE I. THE DISTRIBUTION OF PRIME NUMBERS IN 1 TO 8 CLASSES IN THE GENERALIZED ARTIN CONJECTURE

Interval, ml	**$P_{4.2}$**	**$P_{4.4}$**	**$P_{4.6}$**	**$P_{4.8}$**	**$P_{4.10}$**	**$P_{4.12}$**	**$P_{4.14}$**	**$P_{4.16}$**
0.0–10.0	0.5609	0.0935	0.0997	0.0701	0.0283	0.0166	0.0134	0.0176

TABLE II. THE DISTRIBUTION OF PRIME NUMBERS IN 9 TO 16 CLASSES IN THE GENERALIZED ARTIN CONJECTURE

Interval, ml	**$P_{4.18}$**	**$P_{4.20}$**	**$P_{4.22}$**	**$P_{4.24}$**	**$P_{4.26}$**	**$P_{4.28}$**	**$P_{4.30}$**	**$P_{4.32}$**
0.0–10.0	0.0111	0.0047	0.0051	0.0125	0.0036	0.0022	0.0050	0.0044

TABLE III. THE RESIDUE CLASSES OF RELATIVELY PRIME WITH 16

Interval, ml	**$P_{3.1}$**	**$P_{3.2}$**	**$P_{3.3}$**	**$P_{3.4}$**	**$P_{3.5}$**	**$P_{3.6}$**	**$P_{3.7}$**	**$P_{3.8}$**	**$P_{3.9}$**	**$P_{3.10}$**
0.0 – 1.0	0.3838	0.2827	0.0683	0.0613	0.0094	0.0400	0.0092	0.0154	0.0077	0.0219

An analysis of the data of the tables above shows that for these numbers the Artin conjecture is true on the set of primes $|P| = 10^9$.

CONCLUSIONS

The results of experimental mathematics in table 3 of the first iteration confirm that Artin's conjecture is correct. Estimates of the constants are obtained to within a third decimal place. The data of the table confirm the generalized Artin hypothesis for $a = 4$ and the assumption that $\sum_{i=1}^{\infty} c(4,2i) = 1$. The obtained results are the basis for constructing an analytic proof of the Artin conjecture and its generalization.

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