Parameterized Archimedean Triangular Norm

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Abstract—**The use and construction of triangular norms as the basic part of logical connectives of fuzzy systems is considered. Herewith, the transformation of functions which use the Lambert W-function and its inverse is taken for the basis. To expand the functional characteristics of the constructed new triangular norm, its parametrization is applied. This provided a possible change in the hyper-surface of the triangular norm within wide limits. It is proved that the proposed function satisfies the requirements of such axioms as commutativity, associativity, monotonicity and boundary conditions. Proposed parameterized triangular norm is of a strict and Archimedean type.**

Index Terms—**triangular norms, fuzzy logic, logical connectives.**

I. Introduction

The concept of a triangular norm was introduced by Menger [1] for the generalization of the inequality of a triangle in a certain space. The modern definition of the triangular norm and the dual to it conorm were introduced by Schweizer and Sklar in the theory of probabilistic metric spaces [2]. These norms can be used as a generalization of the connectives of Boolean logic to many-valued and fuzzy logic [1, 3-5]. This fact has led to an increase in the interest and comprehensive development of the theory of triangular norms [6]. Nowadays fuzzy sets theory and fuzzy logic are the basis for modern systems of management and decision-making in many branches of industry. Therefore construction of new triangular norms is an actual and modern task.

This paper is organized as follows. We briefly describe classic triangular norms in Section II. New parameterized Archimedean triangular norm and its properties are presented in Section III.

II. Known Logical Connectives

Triangular norms (*T*-norms) and conorms (*S*-norms) are often used as logical connectives in fuzzy logic systems. Let us note that *T*-norm and corresponding *S*-norm are bound by the de Morgan rule. The very first pair of triangular norms were proposed by L. Zade [7] as min and max operators

$$T_{£}(x,y) = \max(x+y-1,0),$$

$$S_{£}(x,y) = \min(x+y,1).$$

They are representatives of the class of conditional triangular norms, to which also belong such norms as Łukasiewicz, Yager, Fodor, Dobois-Prade and others [1].

The second class of triangular norms – algebraic norms – is based on the algebraic strict Archimedean norm:

$$T_a(x,y) = xy,$$
$$S_a(x,y) = x+y-xy.$$

Such norms as Einstein, Hamacher, Dombi etc. [1, 3] also belong to it.

III. New Parametrized Triangular Norm

To expands the functional characteristics of logical connectives, the parameterization of the triangular norms by power transformation of the argument is used. We set the goal to construct new triangular norm the characteristics of which would change due to the presence of an additional parameter in the generator function. By developing the previously constructed by us triangular norms [8] we propose to use for this purpose the function described by the expression

$$T(x,y) = \exp\{-\alpha W[\frac{1}{\alpha}(\frac{(-\ln(x))^{\alpha}}{x} + \frac{(-\ln(y))^{\alpha}}{y})^{\frac{1}{\alpha}}]\} \quad (1)$$

where $W(x)$ – Lambert's function, such that $W^{-1}(x) = xe^x$, $\alpha > 0$, $x, y \in [0,1]$. Let us prove, that function (1) is a triangular norm, that is, it satisfies the requirements of a number of axioms, in particular, it should be commutative, associative, monotone, and satisfy the boundary conditions.

Commutativity.

$$T(x,y) = \exp\{-\alpha W[\frac{1}{\alpha}(\frac{(-\ln(x))^{\alpha}}{x} + \frac{(-\ln(y))^{\alpha}}{y})^{\frac{1}{\alpha}}]\} =$$
$$= \exp\{-\alpha W[\frac{1}{\alpha}(\frac{(-\ln(y))^{\alpha}}{y} + \frac{(-\ln(x))^{\alpha}}{x})^{\frac{1}{\alpha}}]\} =$$
$$= T(y,x)$$

Associativity.

$$T(T(x,y),z)=\exp\{-\alpha W[\frac{1}{\alpha}(\frac{(-\ln(T(x,y)))^\alpha}{T(x,y)}+\frac{(-\ln(z))^\alpha}{z})^{\frac{1}{\alpha}}]\}=$$

$$=\exp\{W[(\frac{(-\ln(\exp(-\alpha W[\frac{1}{\alpha}(\frac{(-\ln(x))^\alpha}{x}+\frac{(-\ln(y))^\alpha}{y})^{\frac{1}{\alpha}}])))^\alpha}{\exp(-\alpha W[\frac{1}{\alpha}(\frac{(-\ln(x))^\alpha}{x}+\frac{(-\ln(y))^\alpha}{y})^{\frac{1}{\alpha}}])}+$$

$$+\frac{(-\ln(z))^\alpha}{z})^{\frac{1}{\alpha}}\times\frac{1}{\alpha}]\times(-\alpha)\}=$$

$$=\exp\{-\alpha W[\frac{1}{\alpha}((\alpha W[\frac{1}{\alpha}(\frac{(-\ln(x))^\alpha}{x}+\frac{(-\ln(y))^\alpha}{y})^{\frac{1}{\alpha}}])^\alpha\times$$

$$\times\exp(\alpha W[\frac{1}{\alpha}(\frac{(-\ln(x))^\alpha}{x}+\frac{(-\ln(y))^\alpha}{y})^{\frac{1}{\alpha}}])+\frac{(-\ln(z))^\alpha}{z})^{\frac{1}{\alpha}}]\}=$$

$$=\exp\{-\alpha W[\frac{1}{\alpha}((\alpha W[\frac{1}{\alpha}(\frac{(-\ln(x))^\alpha}{x}+\frac{(-\ln(y))^\alpha}{y})^{\frac{1}{\alpha}}])^\alpha\times$$

$$\times(\frac{\frac{1}{\alpha}(\frac{(-\ln(x))^\alpha}{x}+\frac{(-\ln(y))^\alpha}{y})^{\frac{1}{\alpha}}}{W[\frac{1}{\alpha}(\frac{(-\ln(x))^\alpha}{x}+\frac{(-\ln(y))^\alpha}{y})^{\frac{1}{\alpha}}]})^\alpha+\frac{(-\ln(z))^\alpha}{z})^{\frac{1}{\alpha}}]\}=$$

$$=\exp\{-\alpha W[\frac{1}{\alpha}(\alpha^\alpha\frac{1}{\alpha^\alpha}(\frac{(-\ln(x))^\alpha}{x}+$$

$$+\frac{(-\ln(y))^\alpha}{y}+\frac{(-\ln(z))^\alpha}{z}))^{\frac{1}{\alpha}}]\}=$$

$$=\exp\{-\alpha W[\frac{1}{\alpha}(\frac{(-\ln(x))^\alpha}{x}+\frac{(-\ln(y))^\alpha}{y}+\frac{(-\ln(z))^\alpha}{z})^{\frac{1}{\alpha}}]\}. \quad (2)$$

On the other side

$$T(x,T(y,z))=\exp\{-\alpha W[\frac{1}{\alpha}(\frac{(-\ln(x))^\alpha}{x}+\frac{(-\ln(T(y,z)))^\alpha}{T(y,z)})^{\frac{1}{\alpha}}]\}=$$

$$=\exp\{-\alpha W[\frac{1}{\alpha}(\frac{(-\ln(x))^\alpha}{x}+$$

$$+\frac{(-\ln(\exp(-\alpha W[\frac{1}{\alpha}(\frac{(-\ln(y))^\alpha}{y}+\frac{(-\ln(z))^\alpha}{z})^{\frac{1}{\alpha}}])))^\alpha}{\exp(-\alpha W[\frac{1}{\alpha}(\frac{(-\ln(y))^\alpha}{y}+\frac{(-\ln(z))^\alpha}{z})^{\frac{1}{\alpha}}])})^{\frac{1}{\alpha}}]\}=$$

$$=\exp\{-\alpha W[\frac{1}{\alpha}((\alpha W[\frac{1}{\alpha}(\frac{(-\ln(x))^\alpha}{x}+(\alpha W[\frac{1}{\alpha}(\frac{(-\ln(y))^\alpha}{y}+$$

$$+\frac{(-\ln(z))^\alpha}{z})^{\frac{1}{\alpha}}])^\alpha\exp(\alpha W[\frac{1}{\alpha}(\frac{(-\ln(y))^\alpha}{y}+\frac{(-\ln(z))^\alpha}{z})^{\frac{1}{\alpha}}]))^{\frac{1}{\alpha}}]\}=$$

$$=\exp\{-\alpha W[\frac{1}{\alpha}((\alpha W[\frac{1}{\alpha}(\frac{(-\ln(x))^\alpha}{x}+(\alpha W[\frac{1}{\alpha}(\frac{(-\ln(y))^\alpha}{y}+$$

$$+\frac{(-\ln(z))^\alpha}{z})^{\frac{1}{\alpha}}])^\alpha(\frac{\frac{1}{\alpha}(\frac{(-\ln(y))^\alpha}{y}+\frac{(-\ln(z))^\alpha}{z})^{\frac{1}{\alpha}}}{W[\frac{1}{\alpha}(\frac{(-\ln(y))^\alpha}{y}+\frac{(-\ln(z))^\alpha}{z})^{\frac{1}{\alpha}}]})^\alpha)^{\frac{1}{\alpha}}]\}=$$

$$=\exp\{-\alpha W[\frac{1}{\alpha}(\frac{(-\ln(x))^\alpha}{x}+\alpha^\alpha\frac{1}{\alpha^\alpha}(\frac{(-\ln(y))^\alpha}{y}+$$

$$+\frac{(-\ln(z))^\alpha}{z}))^{\frac{1}{\alpha}}]\}=\exp\{-\alpha W[\frac{1}{\alpha}(\frac{(-\ln(x))^\alpha}{x}+$$

$$+\frac{(-\ln(y))^\alpha}{y}+\frac{(-\ln(z))^\alpha}{z})^{\frac{1}{\alpha}}]\}. \quad (3)$$

Since the obtained expressions (2) and (3) are the same, then $T(T(x,y),z)=T(x,T(y,z))$.

Boundary conditions: $T(x,1)=x$.

$$T(x,y)=\exp\{-\alpha W[\frac{1}{\alpha}(\frac{(-\ln(x))^\alpha}{x}+\frac{(-\ln(y))^\alpha}{y})^{\frac{1}{\alpha}}]\}$$

and at $y=1$

$$T(x,y)=\exp\{-\alpha W[\frac{1}{\alpha}(\frac{(-\ln(x))^\alpha}{x}+\frac{(-\ln(1))^\alpha}{1})^{\frac{1}{\alpha}}]\}=$$

$$=\exp\{-\alpha W[\frac{1}{\alpha}(\frac{(-\ln(x))^\alpha}{x})^{\frac{1}{\alpha}}]\}=\exp\{-\alpha W[\frac{1}{\alpha}\frac{(-\ln(x))}{x^{1/\alpha}}]\}. \quad (4)$$

Now let's prove that

$$\alpha W[\frac{1}{\alpha}(\frac{(-\ln(x))}{x^{1/\alpha}})]=-\ln(x). \quad (5)$$

From the expression (5) we will obtain

$$W[\frac{1}{\alpha}\frac{(-\ln(x))}{x^{1/\alpha}}]=-\frac{1}{\alpha}\ln(x). \quad (6)$$

Taking the inverse function $W^{-1}(\cdot)$ from each side of equality (6) we obtain

$$\frac{1}{\alpha}\cdot\frac{(-\ln(x))}{x^{1/\alpha}}=(-\frac{1}{\alpha}\ln(x))\exp[-\frac{1}{\alpha}\ln(x)],$$

where do we get from

$$-\frac{\ln(x)}{\alpha x^{1/\alpha}}=-\frac{1}{\alpha}\ln(x)\exp[-\ln(x^{1/\alpha})]=-\frac{\ln(x)}{\alpha x^{1/\alpha}},$$

that means that equality (5) is true. Then substituting (5) in (4) we get

$$T(x,1)=\exp(-(\ln(x)))\equiv x,$$

i.e. $T(x,1)=x$.

Monotonicity. Let $z\geq y$. Then

$$\frac{(-\ln(z))^\alpha}{z}\leq\frac{(-\ln(y))^\alpha}{y}. \quad (7)$$

Let us add to both parts of the inequality (7) $(-\ln(x))^{\alpha}/x$, getting

$$\frac{(-\ln(x))^{\alpha}}{x}+\frac{(-\ln(z))^{\alpha}}{z}\le\frac{(-\ln(x))^{\alpha}}{x}+\frac{(-\ln(y))^{\alpha}}{y} \quad (8)$$

We raise each sides of the inequality (8) to the degree $1/\alpha$

$$(\frac{(-\ln(x))^{\alpha}}{x}+\frac{(-\ln(z))^{\alpha}}{z})^{\frac{1}{\alpha}}\le(\frac{(-\ln(x))^{\alpha}}{x}+\frac{(-\ln(y))^{\alpha}}{y})^{\frac{1}{\alpha}} \quad (9)$$

and multiply them by $1/\alpha$:

$$\frac{1}{\alpha}(\frac{(-\ln(x))^{\alpha}}{x}+\frac{(-\ln(z))^{\alpha}}{z})^{\frac{1}{\alpha}}\le \le\frac{1}{\alpha}(\frac{(-\ln(x))^{\alpha}}{x}+\frac{(-\ln(y))^{\alpha}}{y})^{\frac{1}{\alpha}}. \quad (10)$$

Since the function $W(x)$ is monotonically increasing, inequality (10) will take the form

$$W[\frac{1}{\alpha}(\frac{(-\ln(x))^{\alpha}}{x}+\frac{(-\ln(z))^{\alpha}}{z})^{\frac{1}{\alpha}}]\le \le W[\frac{1}{\alpha}(\frac{(-\ln(x))^{\alpha}}{x}+\frac{(-\ln(y))^{\alpha}}{y})^{\frac{1}{\alpha}}]. \quad (11)$$

and after multiplying each part of inequality (11) into $(-\alpha)$ we have

$$-\alpha W[\frac{1}{\alpha}(\frac{(-\ln(x))^{\alpha}}{x}+\frac{(-\ln(z))^{\alpha}}{z})^{\frac{1}{\alpha}}]\ge \ge-\alpha W[\frac{1}{\alpha}(\frac{(-\ln(x))^{\alpha}}{x}+\frac{(-\ln(y))^{\alpha}}{y})^{\frac{1}{\alpha}}]. \quad (12)$$

Taking the exponent from each part of the inequality we get

$$T(x,z)=\exp\{-\alpha W[\frac{1}{\alpha}(\frac{(-\ln(x))^{\alpha}}{x}+\frac{(-\ln(z))^{\alpha}}{z})^{\frac{1}{\alpha}}]\}\ge T(x,y)=$$

$$=\exp\{-\alpha W[\frac{1}{\alpha}(\frac{(-\ln(x))^{\alpha}}{x}+\frac{(-\ln(y))^{\alpha}}{y})^{\frac{1}{\alpha}}]\}.$$

Therefore $T(x,z)\ge T(x,y)$.

Consequently, the constructed function meets the axiom requirements to triangular norms.

We will analyze the triangular norm $T(x,y)$ (1) with $x=y$. Then it will take a look

$$T(x,x)=\exp\{-\alpha W[\frac{1}{\alpha}(\frac{(-\ln(x))^{\alpha}}{x}+\frac{(-\ln(x))^{\alpha}}{x})^{\frac{1}{\alpha}}]\}=$$

$$=\exp\{-\alpha W[\frac{2(-\ln(x))}{\alpha x^{1/\alpha}}]\}.$$

Next we show that for $x\in(0,1)$ at $\alpha>0$

$$\exp\{-\alpha W[\frac{2(-\ln(x))}{\alpha x^{1/\alpha}}]\}<x \quad (13)$$

To prove, we will write inequality

$$-2\ln(x)>-\ln(x). \quad (14)$$

Dividing the two parts of the inequality (14) by $\alpha x^{1/\alpha}$, we obtain

$$-\frac{2\ln(x)}{\alpha x^{1/\alpha}}>-\frac{\ln(x)}{\alpha x^{1/\alpha}}. \quad (15)$$

Taking the Lambert function from each part of the inequality (15) we obtain

$$W[-\frac{2\ln(x)}{\alpha x^{1/\alpha}}]>-\frac{1}{\alpha}\ln(x),$$

wherefrom

$$-\alpha W[-\frac{2\ln(x)}{\alpha x^{1/\alpha}}]<\ln(x) \text{ and } \exp\{-\alpha W[-\frac{2\ln(x)}{\alpha x^{1/\alpha}}]\}<x.$$

Therefore $T(x,x)<x$.

IV. CONCLUSION

A new parameterized Archimedean triangular norm is presented. Such constructed triangular norm expands the functional characteristics of logical connectives in fuzzy logic due to the presence of an additional parameter in the generator function. Thus, this triangular norm is more flexible and can be used as logical connective in fuzzy logic decision-making systems to increase they effectiveness and extend functionality.

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