Application of the Thurstone Model

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Abstract—**Thurstone model was analized evaluation of expert information. The inputs for evaluation are matrices of pairwise comparisons created by experts in the evaluated area. To solve the received after processing input data of the system of equations, the method of least squares is used. After solving the system of equations, the mathematical expectation for each of the evaluated objects is obtained, the higher the mathematical expectation, the better object according to the given criteria in the opinion of the polled experts.**

Index Terms—**Thurstone; analize; decision theory**

I. Introduction

Methods of expert assessments are now used in situations where choices, justifications and evaluations of the consequences of decisions can not be made on the basis of accurate calculations. Such situations often arise in the development of modern problems in the management of public production and, especially, in forecasting and long-term planning. Today, expert assessments are widely used in socio-political and scientific and technical forecasting, in the planning of the national economy, industries, associations, in the development of large scientific and technical, economic and social programs, in the adoption of separate management problems.

In the course of the development of social production, not only the complexity of management, but also the requirements for the quality of decisions are growing. In order to increase the validity of decisions and to take into account the numerous criteria influencing the decision, a comprehensive analysis is required, based both on the calculations and on the reasoned opinions of managers and specialists who are familiar with the state of affairs and prospects of development in various areas of practical activity. The application of expert methods ensures the active and purposeful participation of specialists at all stages of decision-making, which allows them to significantly improve their quality and effectiveness.

When we are solving economic and production problems often encounter the need for formalization of systems characterized by a large number of criteria. At the same time from the criteria can be allocated such that can not be measured and expressed in mathematical units. These include the responsibility of the supplier, the prestige of the firm, the reliability of the partner, the buyer's preferences in choosing the product, the opinion of the population about the methods of work of local authorities and others. At the same time, there is a class of parametric factors that can be measured in one or another unit, but sometimes measuring these factors takes a long time and high costs for researching the system. In such cases, it is advisable to resort to the opinion of experts - specialists whose experience in the studied area can significantly increase the amount of available information about the object.

Methods of accounting for factors that are not parameterized in the study of systems, or are parameterized, but difficult to measure, are called expert appraisal methods. These methods allow to objectively procesing the qualitative data obtained as a result of expert research, questioning, testing and other methods.

The purpose of the work is to process expert estimates using the method of Thurstone, which is considered one of the most precise in the theory of decision-making.

II. The Algorithm of the Model of Pairwise Comparisons of Terstoun

A. Normal distribution of ratings

So, we have an objects $a_1, a_2, ..., a_n$ and experts $r_1, r_2, ..., r_N$. Assume that the opinion of one expert r_i, of one object a_j (i - any number in the set of 1, 2, ..., N; j is any number in the set of 1, 2, ..., n) is a normal distribution.

Simply put, this means that for all surveys conducted in different conditions, the most commonly used is the value of m_{ij} (mathematical expectation, that is, the average normal distribution value), and more rarely - other estimates. And the further, the number is placed from m_{ij}, the less likely it will be encountered as such.

Probably the natural proposition seems to be the correct estimate of the opinion of our expert about the object corresponding mathematical expectation.

It turns out that the dispersion of distributions can also be interpreted in a natural way (recall that the normal distribution is uniquely determined by the values of the mathematical expectation and dispersion or the mean square of the deviation).

It is easy to understand that the variance speaks of the degree of confidence (conviction) of an expert in his opinion about this object. It is clear that this degree of confidence can be dependent on various factors: the nature (principle) of the expert, his knowledge of the evaluated objects, the importance of these objects for the expert, etc. So far, we assume that the variances of those distributions that correspond to the views of one expert on different objects are different.

B. Construction of a system of equations for finding the values of the scale of the objects being studied

Consequently, we want to find average values (mathematical expectations) for some hypothetical random variables. The distributions corresponding to these quantities are unknown to us and we can hardly find them, calculate experimentally. So we have to go the other way. Let's recall something about the concept of probabilistic distribution.

The normal distribution in mathematical statistics is well studied. This, in particular, means that for this distribution there are statistical tables that allow for each value of a random variable to find the probability of its occurrence, for each probability-value, which are probable from this probability. We need to find certain values of our random variables (those that are average). So, we should try to analyze which probabilities we can rely on. To understand how PCP (**Pairwise Comparison Process**) matrices can be associated with some possibilities, consider another element of the model that was proposed by Terstoun.

First of all, we note that, since for each object the plurality of evaluations of different experts corresponds to the same random variable, it is logical to assume that an approximate distribution of this value can be found in two ways: by repeatedly interviewing one (any) expert, or through a one-time survey of many experts. The result will be the same.

Now let's put all our matrices of PCP. It is easy to understand that then at the intersection of the *i*-th row and the *j*-th column of the resulting matrix-sum there will be an amount of experts who say that $a_i > a_j$ (δ=1) if $a_i < a_j$ (δ=0). Divide this amount into a total number of experts and get the corresponding share. Denote it via p_{ij}:

$$p_{ij} = \frac{1}{N}\sum_{l}\delta_{ij}^{l} \tag{1}$$

Following the above logic, which allows us to replace the set of opinions of various experts by the repeated opinion of one expert, we will assume that p_{ij} speaks of how often one expert will prefer the *i*-th object *j*-th (if you imagine that we many times we put the expert all the pairs of objects examined).

Note that the matrix $\| p_{ij} \|$ has a number of properties, the knowledge of which can help in the use of the description of the theoretical described provisions in practice, namely, for all *i* and *j* ratio: $0 < p_{ij} <1$; $p_{ij} + p_{ji} = 1$.

Now let us recall the law of comparison of Thurstone: the more often, when multiple surveys some expert will prefer the object a_i, and not the object a_j, the further placed the estimates of this expert on the scale of the values of the objects under consideration. Probably, given that any expert corresponds to a set of values of the scale, each of which meets a certain probability (that is, for each expert corresponds to some random variable), it is natural to assume that the proportion p_{ij} equals the probability that our *i*-th accidental the value (that is, the random variable corresponding to the *i*-th object) is greater than *j*-th (corresponding to the *j*-th object), or, in the formal language:

$$p_{ij} = P\left(\xi_i > \xi_j\right) \tag{2}$$

Consequently, our empirical data (the total matrix of PCP) give us some sort of probability. In order to become clear, how can we, by using the knowledge of these probabilities and

TABLE I. STANDARDIZED NORMAL DISTRIBUTION TABLE

P, %	99.99	99.90	99.00	97.72	97.50
m/D^2	3.715	3.090	2.326	2.000	1.960

tables for normal distribution, turn to the mean values of the random variables i and j, introduce a new notation: $\xi_{ij} = \xi_i - \xi_j$

Then, the expression for pij will be rewritten in the form:

$$p_{ij} = P\left(\xi_{ij} > 0\right) \tag{3}$$

The following relationships are based on known results from the field of mathematical statistics. They do not use any models of perception, no assumptions about the essence of what is happening in the minds of one expert, the relationship of processes occurring in the representations of various experts, etc. The Element ξ_{ij}, being the difference between two normally distributed random variables ξ_i and ξ_j with mathematical expectations mi and mj with mean square deviations σ_i and σ_j respectively, is also a normally distributed random variable with mathematical expectation mij and mean square deviation σ_{ij}, which are defined as follows:

$$m_{ij} = m_i - m_j \tag{4}$$

$$\sigma_{ij} = \sigma_i^2 + \sigma_j^2 - 2r_{ij}\sigma_i\sigma_j \tag{5}$$

where r_{ij} is the correlation coefficient between ξ_i and ξ_j.

On the basis of the ratio "(3)" we can, by using the tables of normal distribution, find the corresponding value of p_{ij} in the left part of this equality of the probability of the value ξ_{ij} However, to do this, we need some additional considerations. The fact is that known statistical tables are designed only for the so-called standardized normal distribution, that is, for such random variables of the ξ_{stand}, which corresponds to the null average and unit variance (of course, it is not possible to calculate the tables for all existing normal distributions, since as mathematical expectations can act any valid numbers, and as a dispersion, any positive real numbers). Nevertheless, the table "(1)" can nonetheless be useful if we use the position known in mathematical statistics:

$$P\left(\xi_{ij} > 0\right) = P\left(\xi_{\text{stand}} > \left(\frac{m_{ij}}{\sigma_{ij}}\right)\right) \tag{6}$$

Also, using the table "(1)" for a standardized normal distribution, the value of m_{ij}/σ_{ij} can be found on the basis of the ratio "(6)". Let's denote it through z_{ij}. It is clear that $m_{ij} = \sigma_{ij} \cdot z_{ij}$, which by virtue of "(4)" and "(5)" are equivalent to the equation:

$$m_i - m_j = \sqrt{z_{ij}\left(\sigma_i^2 + \sigma_j^2 - 2r_{ij}\sigma_i\sigma_j\right)} \qquad (7)$$

We obtained a system of equations for finding the desired scale values m_i and m_j (i and j were arbitrary numbers of our objects, therefore equations of type "(7)" we will have as many pairs of these objects can be folded). Let's emphasize, that the equation (7) "proceeds from the total matrix of the PCP very quickly: for each frequency p_{ij} immediately, only by looking in the corresponding statistical table, we find z_{ij} and, therefore, the equations themselves. Solving the system of equations Let's start with the fact that in addition to the values of the scale of the objects under study, the system "(7)" contains other unknowns: σ_i, σ_j, r_{ij}. Let's do with them just as Terstoun and his followers did. First of all, we simplify the equation (7) by making some additional assumptions about the properties of our models, which are related to the values of r_{ij}, σ_i, σ_j. We note that different ways of such simplification are known in the literature. Different constraints on these parameters correspond to different models. That is why at the beginning of this section the model of the Thurstone is written. So, we can make the following assumptions: First, assume that $r_{ij} = 0$, which essentially facilitates the solution of the system "(7)", since on the right side of this system one term disappears with this assumption. But it is important for us to understand what changes in our model make this assumption. Let us recall that r_{ij} is the correlation coefficient between two random variables: i and j. It is easy to understand that the presence of an appropriate connection means the dependence of the expert's opinion on the i-th object from his thoughts on the j-th object. And our assumption denies this dependence. Equating to zero the considered correlation coefficient, we impose and corresponding restrictions on our model. Secondly, we assume that $\sigma_i = \sigma_j = \sigma$. In other words, assume that the measure of confidence in the estimates by our experts of different objects is the same. Of course, this assumption is more questionable than the assumption made above that different experts have the same measure of confidence in the evaluation of the same object. Consequently, the system "(7)" follows from the assumptions made:

$$m_i - m_j = \sqrt{2z_{ij}\sigma} \qquad (8)$$

In this system, besides the desired quantities m_1, m_2, ..., m_n is another unknown - σ. It is possible to find it only by experimentally studying the distribution of expert estimates of all of the objects under consideration. Therefore, we assume that σ = 1. But let us not forget that the solution, whatever it may be, will always be such that the difference (m_i-m_j) is determined only with the accuracy of a constant constant-σ. The degree of ambiguity determines the type of scale. In this case, this degree (that is, the difference between the scale values is determined to the constant of the constant factor) indicates that we are dealing with the interval scale. If any set of numbers will be the solution of our system, then the same solution will be any other set of numbers, which proceeds from the first way of stretching (compressing) all the intervals between them in the same number of times. So, inter alia, σ = 1 and let us turn to the discussion of the solution of the system "(8)." Firstly, the system under consideration is redefined - the number of equations, much larger than the number of unknowns (the number of pairs we can compile from any objects more than the number of objects if we deal with more than three objects). Consequently, this system will most likely not have a solution: even if we find solutions to several equations, it is not necessarily they will satisfy the rest of the equation. Therefore, for the solution of the system of equations we use the method of least squares. More specifically, we will look for such mi and mj, which draw at least the sum of the squares of differences between the right and left parts of the system "(8)".

III. Conclusions

Therefore, on the basis of the above considerations, the application decision support system based on Thurstone model can be designed.

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